

# Scheme Dependence and Multiple Couplings

Ian Jack<sup>a</sup> and Hugh Osborn<sup>b</sup>

<sup>a</sup>Department of Mathematical Sciences, University of Liverpool,  
Liverpool L69 3BX

<sup>b</sup>Department of Applied Mathematics and Theoretical Physics, Wilberforce Road,  
Cambridge CB3 0WA

## Abstract

For theories with multiple couplings the perturbative  $\beta$ -functions for scalar, Yukawa couplings are expressible in terms of contributions corresponding to one particle irreducible graphs and also contributions which are one particle reducible depending on the anomalous dimension. Here we discuss redefinitions, or changes of scheme, which preserve this structure. The redefinitions allow for IPR contributions of a specific form, as is necessary to encompass the relation between  $\overline{\text{MS}}$  and momentum subtraction renormalisation schemes. Many multiply 1PR terms in the transformed  $\beta$ -function are generated but these can all be absorbed into antisymmetric contributions to the anomalous dimensions which are essentially arbitrary and can be discarded. As an illustration the results are applied to the scheme dependence of the anomalous dimension, which determines the  $\beta$ -function, for  $\mathcal{N} = 1$  supersymmetric scalar fermion theories in four dimensions up to four loops.

# 1 Introduction

No physical quantities depend on the choice of regularisation scheme. As is of course well known different schemes are related by reparameterisations of the finite couplings specifying the renormalised theory. The  $\beta$ -functions, determining the RG flow of the couplings under changes of scale, then transform as vector fields under such reparameterisations.

However in perturbative calculations the form of the  $\beta$ -functions are not entirely arbitrary vector fields. The counterterms which are added to the original Lagrangian to ensure finiteness order by order are determined entirely by considerations of one particle irreducible, 1PI, graphs whose superficial degree of divergence is zero or higher. The amplitudes corresponding to one particle reducible, 1PR, graphs are then automatically finite. Although the local divergent terms which are subtracted by the counterterms are 1PI the rescaling of the fields necessary to ensure a canonical normalisation of the kinetic term, and which is essential in the derivation of RG equations, ensure that the  $\beta$ -functions for scalar or Yukawa couplings have 1PR pieces. These 1PR contributions have a very specific form, being determined by the anomalous dimension.

In this paper we address the question of what redefinitions of the couplings are allowed which preserve the perturbative structure of the  $\beta$ -functions. Manifestly for a single coupling there is no way to distinguish 1PI and 1PR contributions to the  $\beta$ -function. Nevertheless for arbitrary couplings each term in the perturbative  $\beta$ -function can be identified with particular 1PI and 1PR graphs. The particular form of the 1PR graphs restricts the resulting form for the perturbative  $\beta$ -functions.

These issues recently came to the fore in a discussion of renormalisable six dimensional  $\phi^3$  theory, with a general interaction  $\frac{1}{6} g_{ijk} \phi_i \phi_j \phi_k$ , [1]. In order to relate results for the  $\beta$ -function when calculated in  $\overline{\text{MS}}$  and momentum subtraction schemes it was crucial to consider 1PR terms in the necessary reparameterisation of the couplings. For the perturbative structure of the  $\beta$ -functions to be preserved it was necessary to restrict the 1PR contributions to the redefinition of the couplings to a particular form and to allow potential antisymmetric 1PR contributions to the anomalous dimension. Here we discuss such issues in general and show how the antisymmetric pieces, which contain 1PR terms, can be dropped and the resulting anomalous dimension is expressible solely in terms of 1PI graphs with vertices determined by the transformed coupling.

As an example we consider  $\mathcal{N} = 1$  supersymmetric scalar fermion theories where due to supersymmetric non renormalisation theorems the  $\beta$ -function is determined by the anomalous dimension. This structure is not preserved by general redefinitions of the couplings but with the results here we are able to identify scheme changes which preserve the supersymmetric form. We are then able to determine the scheme independent contributions at three and four loops. In some cases these may be determined in terms of lower order results using the  $a$ -theorem.

In the following section the requirements for changes of scheme which preserve the structure of the  $\beta$ -function are discussed for both real and complex couplings and then these results are supplied to the supersymmetric case in section 3.

## 2 Changes of Scheme

For theories with multi-component fields  $\phi_i$  and couplings  $g^I$  the  $\beta$ -functions determined by perturbative expansions have the generic form, with  $\gamma_i^j(g)$  the anomalous dimension matrix,

$$\beta^I(g) = \tilde{\beta}^I(g) + (g \gamma(g))^I, \quad (2.1)$$

where  $\tilde{\beta}^I$  and  $\gamma_i^j$  are constructed from 1PI graphs.

Initially we focus on real fields  $\phi_i$  and associated real couplings  $g^I = \{g_{i_1 \dots i_n}\}$ , taking then  $\gamma_i^j \rightarrow \gamma_{ij}$ . In this case in (2.1)

$$(g \gamma(g))_{i_1 \dots i_n} = g_{ji_2 \dots i_n} \gamma_{ji_1}(g) + \dots + g_{i \dots i_{n-1} j} \gamma_{ji_n}(g). \quad (2.2)$$

Acting on the effective action RG flow is generated by

$$\mathcal{D} = \mathcal{D}_\beta - \int (\gamma(g) \phi)_i \frac{\delta}{\delta \phi_i}, \quad (2.3)$$

for

$$\mathcal{D}_\beta = \beta^I(g) \frac{\partial}{\partial g^I} = (\tilde{\beta}(g) + (g \gamma(g)))^I \frac{\partial}{\partial g^I}. \quad (2.4)$$

Crucially there is an arbitrariness under

$$\gamma(g) \rightarrow \gamma(g) + \omega, \quad \omega^T = -\omega, \quad (2.5)$$

since the effective action is invariant under the action of  $\mathcal{D}$  for arbitrary choices of  $\omega$ . If  $\omega$  is the generator of a symmetry of the theory then the couplings are constrained by  $(g \omega)^I = 0$ . In consequence, in the context of the RG flow generated by  $\mathcal{D}$ ,  $\gamma(g)$  may always be chosen to be symmetric.

We first consider an infinitesimal change in the couplings  $\delta g^I$  for which

$$\delta \beta^I = \beta^J \frac{\partial}{\partial g^J} \delta g^I - \delta g^J \frac{\partial}{\partial g^J} \beta^I, \quad (2.6)$$

and assume changes such that

$$\delta g^I(g) = f^I(g) + (g c(g))^I, \quad (2.7)$$

where  $f^I(g)$ ,  $c_{ij}(g)$  correspond to one particle irreducible (1PI) graphs with  $n$ , 2 external lines. Initially we do not impose any requirements for  $c, \gamma$  to be symmetric. The variation (2.6) gives

$$\delta \beta^I(g) = \delta \tilde{\beta}^I(g) + (g \delta \gamma(g))^I + (g \omega(g))^I, \quad \omega(g) = \gamma(g)^T c(g) - c(g)^T \gamma(g), \quad (2.8)$$

with

$$\begin{aligned} \delta \tilde{\beta}^I(g) &= (\tilde{\beta}(g) + (g \gamma(g)))^J \frac{\partial}{\partial g^J} f^I(g) - (f(g) + (g c(g)))^J \frac{\partial}{\partial g^J} \tilde{\beta}^I(g) \\ &\quad - (f(g) \gamma(g))^I + (\tilde{\beta}(g) c(g))^I, \end{aligned} \quad (2.9)$$

and

$$\begin{aligned}\delta\gamma(g) &= (\tilde{\beta}(g) + (g\gamma(g)))^J \frac{\partial}{\partial g^J} c(g) - (f(g) + (g c(g)))^J \frac{\partial}{\partial g^J} \gamma(g) \\ &\quad + [\gamma(g), c(g)] - \omega(g).\end{aligned}\tag{2.10}$$

Crucially, with the assumptions on the form of  $f^I$ ,  $c$ ,  $\delta\tilde{\beta}^I$ ,  $\delta\gamma$  correspond just to the contributions of 1PI graphs (the second lines of (2.9), (2.10) remove 1PR terms arising from the differentiation in the first line acting on the couplings which couple to the external lines). In particular  $(\gamma(g)g)^J \partial_J c(g) - c(g)\gamma(g) - \gamma(g)^T c(g)$  is 1PI so long as  $c$  is 1PI. A similar result holds for  $c \leftrightarrow \gamma$ . If  $c, \gamma$  are symmetric then so is  $\delta\gamma$  and the last line of (2.10) vanishes.

The extension of (2.8) to finite redefinitions of the couplings, or general scheme changes, requires the  $\beta$ -function to transform as a tangent vector to the manifold parameterised by the couplings and hence

$$\beta'^I(g') = \beta^J(g) \frac{\partial}{\partial g^J} g'^I(g) \quad \text{for } g^I \rightarrow g'^I(g).\tag{2.11}$$

In perturbation theory  $g'^I$  is required to have a power series expansion in  $g$  and to lowest order  $g'^I(g) = g^I + \mathcal{O}(g^2)$ . For redefinitions which preserve the structure (2.1), it is necessary that the transformed  $\beta$ -function has the form

$$\beta'^I(g') = \tilde{\beta}'^I(g') + (g'(\gamma'(g') + \Omega))^I, \quad \Omega^T = -\Omega,\tag{2.12}$$

with  $\tilde{\beta}'^I(g')$ ,  $\gamma'(g')$  expressible in terms of 1PI contributions while  $\Omega$  is expressed in terms of 1PR graphs. We define first  $(g \circ C)^I$  where

$$(g \circ C)_{i_1 \dots i_n} = g_{j_1 \dots j_n} C_{j_1 i_1} \dots C_{j_n i_n}.\tag{2.13}$$

If  $C = \mathbb{1} + c + \dots$  then  $(g \circ C)^I = g^I + (gc)^I + \dots$ . We then consider changes of scheme of the form

$$g'^I = ((g + f(g)) \circ C(g))^I,\tag{2.14}$$

where  $f^I(g)$  is expressed in terms of 1PI graphs with  $n$  external lines.  $C$  is also determined in terms of 1PI graphs later,  $(g + f) \circ C$  generates 1PR terms in the redefinition of the couplings. Using (2.14) in (2.11) we obtain

$$\begin{aligned}\tilde{\beta}'^I(g') &= (\tilde{\beta}_f(g + f) \circ C)^I, \\ \tilde{\beta}_f^I(g + f) &= \tilde{\beta}^I(g) + \mathcal{D}_\beta f^I(g) - (f(g)\gamma(g))^I.\end{aligned}\tag{2.15}$$

$\tilde{\beta}_f^I(g)$  is expressible as a sum of 1PI contributions as a function of  $g$ ,  $\tilde{\beta}_f(g + f) \circ C$  ensures that  $\tilde{\beta}'^I(g')$  is also given in terms of 1PI graphs as a function of  $g'$ . For  $C = \mathbb{1} + c$  and  $c, f$  infinitesimal (2.15) reduces to (2.9) for  $\delta\tilde{\beta}^I(g) = \tilde{\beta}'^I(g) - \tilde{\beta}^I(g)$ . Furthermore (2.12) is then satisfied if we take

$$\gamma'(g') + \Omega = C^{-1}(\mathcal{D}_\beta + \gamma(g))C.\tag{2.16}$$

As argued above any antisymmetric  $\Omega$  is irrelevant when considering the action of  $\mathcal{D}$  in (2.3).

To analyse this further we assume  $\gamma^T = \gamma$  and

$$\begin{aligned} C(g) &= \mathbb{1} + c(g) + a_2 c(g)^2 + a_3 c(g)^3 + \dots, \quad c^T = c, \\ C(g)^{-1} &= \mathbb{1} - c(g) - (a_2 - 1) c(g)^2 - (a_3 - 2a_2 + 1) c(g)^3 + \dots \end{aligned} \quad (2.17)$$

with  $c(g)$  expressible in terms of a sum of contributions corresponding to 1PI graphs. We also define

$$\mathcal{D}_\beta c(g) = \gamma(g) c(g) + c(g) \gamma(g) + c_\beta(g), \quad (2.18)$$

where  $c_\beta(g)$  is expressible in terms of 1PI graphs if  $c(g)$  is 1PI. From (2.16) we then have

$$\begin{aligned} &C(g)^{-1}(\gamma'(g') + \Omega)C(g)^{-1} \\ &= \gamma(g) + c_\beta(g) + \gamma(g) c(g) - c(g) \gamma(g) \\ &\quad + (a_2 - 1)(\gamma(g) c(g)^2 - c(g)^2 \gamma(g)) + (2a_2 - 3) c(g) \gamma(g) c(g) \\ &\quad + (a_2 - 1) c_\beta(g) c(g) + (a_2 - 2) c(g) c_\beta(g) \\ &\quad + (a_3 - 2a_2 + 1)(\gamma(g) c(g)^3 + c_\beta(g) c(g)^2 - c(g)^3 \gamma(g)) \\ &\quad + (2a_3 - 5a_2 + 3) c(g) \gamma(g) c(g)^2 + (2a_3 - 7a_2 + 5) c(g)^2 \gamma(g) c(g) \\ &\quad + (a_3 - 3a_2 + 2) c(g) c_\beta(g) c(g) + (a_3 - 4a_2 + 3) c(g)^2 c_\beta(g) + \mathcal{O}(c^4). \end{aligned} \quad (2.19)$$

Choosing

$$a_2 = \frac{3}{2}, \quad a_3 = \frac{5}{2}, \quad (2.20)$$

gives

$$\begin{aligned} &C(g)^{-1}(\gamma'(g') + \Omega)C(g)^{-1} \\ &= \gamma(g) + c_\beta(g) + \gamma(g) c(g) - c(g) \gamma(g) \\ &\quad + \frac{1}{2}(\gamma(g) c(g)^2 - c(g)^2 \gamma(g) + c_\beta(g) c(g) - c(g) c_\beta(g)) \\ &\quad + \frac{1}{2}(\gamma(g) c(g)^3 - c(g)^3 \gamma(g) + c(g) \gamma(g) c(g)^2 - c(g)^2 \gamma(g) c(g)) \\ &\quad + (c_\beta(g) c(g)^2 - c(g)^2 c_\beta(g)) + \mathcal{O}(c^4). \end{aligned} \quad (2.21)$$

This allows us to identify

$$\gamma'(g') = C(g)(\gamma(g) + c_\beta(g))C(g), \quad (2.22)$$

where, since  $\gamma(g), c_\beta(g)$  correspond to 1PI graphs as functions of  $g$ , then  $\gamma'(g')$  is also given in terms of 1PI graphs as a function of  $g'$ . The remaining part of (2.21) determines  $\Omega = -\Omega^T$  as a sum of 1PR contributions.

To extend these results beyond the first few terms in the expansion of  $C$  it is sufficient to take

$$\begin{aligned} C(g) &= (\mathbb{1} - 2c(g))^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{2}\right)_n (2c)^n, \\ C(g)^{-1} &= (\mathbb{1} - 2c(g))^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{1}{2}\right)_n (2c)^n. \end{aligned} \quad (2.23)$$

Then inserting (2.22) in (2.16), using (2.18) and  $C(g)^{-2} = 1 - 2c(g)$ , gives

$$C(g)^{-1} \Omega C(g)^{-1} = \gamma(g) c(g) - c(g) \gamma(g) + \frac{1}{2} \mathcal{D}_\beta C(g)^{-1} C(g)^{-1} - \frac{1}{2} C(g)^{-1} \mathcal{D}_\beta C(g)^{-1}, \quad (2.24)$$

from which it is evident that, with this choice of  $C(g)$ ,  $\Omega^T = -\Omega$ . Expanding  $\mathcal{D}_\beta C^{-1}$  gives in (2.24)

$$C(g)^{-1} \Omega C(g)^{-1} = \gamma(g) c(g) - c(g) \gamma(g) + \sum_{N=2}^{\infty} \left( \sum_{r=0}^N f_{N,r} c(g)^r \gamma(g) c(g)^{N-r} + \sum_{r=0}^{N-1} g_{N,r} c(g)^r c_\beta(g) c(g)^{N-1-r} \right), \quad (2.25)$$

where, for  $N \geq 2$ ,

$$g_{N,r} = \sum_{m=0}^{N-1-r} h_{N,m}, \quad f_{N,r} = 2g_{N,r-1} - h_{N,r}, \quad h_{N,r} = 2^N \frac{(-\frac{1}{2})_r (-\frac{1}{2})_{N-r}}{r! (N-r)!}. \quad (2.26)$$

and we may take  $g_{N,-1} = 0$ .

If we define

$$c'(g') = -C(g) c(g) C(g), \quad (2.27)$$

then  $c'(g')$  is defined in terms of 1PI graphs so long as  $c(g)$  is, and from (2.23)

$$C'(g') = C(g)^{-1} = (\mathbb{1} - 2c'(g'))^{-\frac{1}{2}}. \quad (2.28)$$

The inverse of (2.14) is then

$$g^I = ((g' + f'(g')) \circ C'(g'))^I, \quad f'(g') = -f(g) \circ C(g), \quad (2.29)$$

with  $f'(g')$  determined by 1PI graphs.

These results may easily be extended to complex fields  $\phi_i, \bar{\phi}^i$  and complex couplings  $g^I = \{g_{i_1 \dots i_p}^{j_1 \dots j_q}\}$ . In this case

$$\beta^I(g) = \tilde{\beta}^I(g) + (g \gamma(g))^I + (\bar{\gamma}(g) g)^I, \quad (2.30)$$

for

$$\begin{aligned} (g \gamma(g))_{i_1 \dots i_p}^{j_1 \dots j_q} &= g_{i_1 \dots i_p}^{k j_2 \dots j_q} \gamma_k^{j_1}(g) + \dots + g_{i_1 \dots i_p}^{j_1 \dots j_{q-1} j_k} \gamma_k^{j_q}(g) \\ (\bar{\gamma}(g) g)_{i_1 \dots i_p}^{j_1 \dots j_q} &= \bar{\gamma}_{i_1}^k(g) g_{k i_2 \dots i_p}^{j_1 \dots j_q} + \dots + \bar{\gamma}_{i_p}^k(g) g_{i_1 \dots i_{p-1} k}^{j_1 \dots j_q}, \end{aligned} \quad (2.31)$$

and  $\bar{\gamma} = \gamma^\dagger$ . In this case (2.3) becomes

$$\mathcal{D} = \mathcal{D}_\beta - \int (\gamma(g) \phi)_i \frac{\delta}{\delta \phi_i} - \int (\bar{\phi} \bar{\gamma}(g))^i \frac{\delta}{\delta \bar{\phi}^i}, \quad (2.32)$$

where  $\gamma \rightarrow \gamma + \omega$ ,  $\bar{\gamma} \rightarrow \bar{\gamma} - \omega$  corresponds to a trivial invariance of the effective action. Since the  $\omega$  terms can be discarded we may impose  $\bar{\gamma} = \gamma$ . To preserve the perturbative structure in (2.31) we may extend (2.14) to

$$g'^I = (C(g) \circ (g + f(g)) \circ C(g))^I, \quad (2.33)$$

with  $f(g)$  representing 1PI contributions to the redefinition of the couplings and  $C_i^j(g)$  expressible as in (2.23) in terms of the 1PI  $c(g) = \bar{c}(g) = c(g)^\dagger$ . In this case (2.15) becomes

$$\begin{aligned} \tilde{\beta}^I(g') &= (C(g) \circ \tilde{\beta}_f(g + f) \circ C(g))^I, \\ \tilde{\beta}_f^I(g + f) &= \tilde{\beta}^I(g) + \mathcal{D}_\beta f^I(g) - (f(g) \gamma(g))^I - (\gamma(g) f(g))^I, \end{aligned} \quad (2.34)$$

and (2.16) becomes

$$\gamma'(g') + \Omega = C(g)^{-1} (\mathcal{D}_\beta + \gamma(g)) C(g), \quad \gamma'(g') - \Omega = (\mathcal{D}_\beta C(g) + C(g) \gamma(g)) C(g)^{-1}, \quad (2.35)$$

with  $\Omega = -\Omega^\dagger$ . Otherwise the discussion follows the same route as before. By acting on the left and the right of (2.35) with  $C^{-1}$  and subtracting the two equations we recover (2.24). The modified  $\gamma'(g')$  is determined by (2.22) and (2.18) as before.

### 3 Supersymmetric Model

A cardinal illustration of the relevance of these issues is the  $\mathcal{N} = 1$  supersymmetric scalar fermion theory in four dimensions. The complex couplings are then symmetric rank three tensors  $Y^{ijk}$ ,  $\bar{Y}_{ijk}$  and there are no 1PI contributions to the  $\beta$ -functions which are determined just by the anomalous dimension  $\gamma_i^j(Y, \bar{Y})$  in the form

$$\beta_Y^{ijk} = (Y \gamma)^{ijk}, \quad \beta_{\bar{Y}}_{ijk} = (\gamma \bar{Y})_{ijk}, \quad (3.1)$$

with notation as in (2.31). Scheme changes at the linearised level preserving (3.1) for this theory were discussed in [2] (see 7.23) and these issues were also considered in [3]. More generally from the results above the form (3.1) is consistent with redefinitions

$$Y'^{ijk} = Y^{lmn} C_l^i C_m^j C_n^k, \quad \bar{Y}'_{ijk} = C_i^l C_j^m C_k^n \bar{Y}_{lmn}, \quad (3.2)$$

where  $C_i^l(Y, \bar{Y})$  is determined in terms of a 1PI  $c_i^l(Y, \bar{Y})$  as in (2.23).

Applying these results first to three loops we let  $Y^{ijk} \rightarrow 4\pi Y^{ijk}$ ,  $\bar{Y}_{ijk} \rightarrow 4\pi \bar{Y}_{ijk}$  and then use a diagrammatic notation where  $Y^{ijk} \rightarrow \text{triple line vertex}$  and  $\bar{Y}_{ijk} \rightarrow \text{triple line vertex}$ . Only lines linking  $Y$  and  $\bar{Y}$  vertices are allowed. At this order

$$\begin{aligned} \gamma(Y, \bar{Y}) &= c_1 \text{ (bubble)} + c_2 \text{ (self-energy)} \\ &+ c_{3A} \text{ (triangle)} + c_{3B} \text{ (triangle)} + c_{3C} \text{ (triangle)} + c_{3D} \text{ (triangle)}. \end{aligned} \quad (3.3)$$

We consider to this order transformations determined by

$$c(Y, \bar{Y}) = \epsilon_1 \text{---} \text{---} \text{---} + \epsilon_2 \text{---} \text{---} \text{---}, \quad (3.4)$$

where correspondingly from (2.27)

$$c'(Y', \bar{Y}') = -\epsilon_1 \text{---} \text{---} \text{---} - (\epsilon_2 - 4\epsilon_1^2) \text{---} \text{---} \text{---}. \quad (3.5)$$

From the definition (2.18)

$$\begin{aligned} c_\beta(Y, \bar{Y}) = & 4c_1\epsilon_1 \text{---} \text{---} \text{---} \\ & + 4c_1\epsilon_2 \text{---} \text{---} \text{---} + 2c_1\epsilon_2 \text{---} \text{---} \text{---} + 4(c_1\epsilon_2 + c_2\epsilon_1) \text{---} \text{---} \text{---}, \end{aligned} \quad (3.6)$$

and then applying the transformation  $Y, \bar{Y} \rightarrow Y', \bar{Y}'$  to  $\gamma(Y, \bar{Y}) + c_\beta(Y, \bar{Y})$  gives according to (2.22)  $\gamma'(Y', \bar{Y}')$  as a sum of 1PI terms of the same form as (3.3) with the one and two loop contributions invariant and at three loops

$$c'_{3A} = c_{3A} + 2X, \quad c'_{3B} = c_{3B} + X, \quad X = -4c_1\epsilon_1^2 + 2c_1\epsilon_2 - 2c_2\epsilon_1, \quad (3.7)$$

with  $c_{3C}, c_{3D}$  also invariant. Calculations [3] give for the scheme invariants to this order

$$c_1 = \frac{1}{2}, \quad c_2 = -\frac{1}{2}, \quad c_{3A} - 2c_{3B} = 0, \quad c_{3C} = 1, \quad c_{3D} = \frac{3}{2}\zeta(3). \quad (3.8)$$

The four loop contributions to the anomalous dimension  $\gamma$  have significantly more terms and it is convenient to separate them into three classes according to the topology of the corresponding diagrams. For planar graphs

$$\begin{aligned} \gamma(Y, \bar{Y})_1^{(4)} = & c_{4A} \text{---} \text{---} \text{---} + c_{4B} \text{---} \text{---} \text{---} \\ & + c_{4C} \left( \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \right) + c_{4D} \text{---} \text{---} \text{---} \\ & + c_{4E} \text{---} \text{---} \text{---} + c_{4F} \text{---} \text{---} \text{---} + c_{4G} \text{---} \text{---} \text{---}, \end{aligned} \quad (3.9)$$

and, with non planar subgraphs,

$$\begin{aligned} \gamma(Y, \bar{Y})_2^{(4)} = & c_{4H} \left( \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \right) \\ & + c_{4I} \text{---} \text{---} \text{---} + c_{4J} \text{---} \text{---} \text{---}, \end{aligned} \quad (3.10)$$



and, with a new topology,

$$\gamma(Y, \bar{Y})_3^{(4)} = c_{4K} \text{---} \langle \text{diagram} \rangle \text{---}. \quad (3.11)$$

At this order (3.4) has additional contributions of the general form

$$c(Y, \bar{Y})^{(3)} = \epsilon_{3A} \text{---} \text{---} + \epsilon_{3B} \text{---} \text{---} + \epsilon_{3C} \text{---} \text{---} + \epsilon_{3D} \text{---} \text{---}, \quad (3.12)$$

while (3.5) is extended by

$$\begin{aligned}
c'(Y', \bar{Y}')^{(3)} = & -(\epsilon_{3A} - 4\epsilon_1\epsilon_2 + 8\epsilon_1^3) \text{---} \text{[Diagram 1]} \text{---} -(\epsilon_{3B} - 2\epsilon_1\epsilon_2 + 4\epsilon_1^3) \text{---} \text{[Diagram 2]} \text{---} \\
& -(\epsilon_{3C} - 8\epsilon_1\epsilon_2 + 16\epsilon_1^3) \text{---} \text{[Diagram 3]} \text{---} -\epsilon_{3D} \text{---} \text{[Diagram 4]} \text{---} .
\end{aligned} \tag{3.13}$$

In this case

$$\begin{aligned}
c_\beta(Y, \bar{Y})_1^{(4)} = & 6 c_1 \epsilon_{3A} \text{---} \text{[diagram: circle with two internal arcs and two external vertices]} \text{---} + 2 c_1 (\epsilon_{3A} + 4 \epsilon_{3B}) \text{---} \text{[diagram: circle with two internal arcs and two external vertices]} \text{---} \\
& + 2(2 c_1 \epsilon_{3A} + c_1 \epsilon_{3C} + c_2 \epsilon_2) \left( \text{---} \text{[diagram: circle with two internal arcs and two external vertices]} \text{---} + \text{---} \text{[diagram: circle with two internal arcs and two external vertices]} \text{---} \right) \\
& + 2(4 c_1 \epsilon_{3B} + c_1 \epsilon_{3C} + c_2 \epsilon_2) \text{---} \text{[diagram: circle with two internal arcs and two external vertices]} \text{---} + 4(c_1 \epsilon_{3C} + c_{3A} \epsilon_1) \text{---} \text{[diagram: circle with two internal arcs and two external vertices]} \text{---} \\
& + 2(c_1 \epsilon_{3C} + 2 c_{3B} \epsilon_1) \text{---} \text{[diagram: circle with two internal arcs and two external vertices]} \text{---} + 4(c_1 \epsilon_{3C} + c_2 \epsilon_2 + c_{3C} \epsilon_1) \text{---} \text{[diagram: circle with two internal arcs and two external vertices]} \text{---},
\end{aligned}
\tag{3.14}$$

and the non planar contributions are

$$c_\beta(Y, \bar{Y})_2^{(4)} = 4 c_1 \epsilon_{3D} \left( \text{diagram 1} + \text{diagram 2} + 2 \text{diagram 3} \right) + 4 c_{3D} \epsilon_1 \text{diagram 4} . \quad (3.15)$$

Transforming  $(Y, \bar{Y}) \rightarrow (Y', \bar{Y}')$  now gives  $\gamma'(Y', \bar{Y}')$ <sup>(4)</sup> of the form (3.9) with

$$\begin{aligned}
c'_{4A} &= c_{4A} + 32 c_1 \epsilon_1^3 - 24 c_1 \epsilon_1 \epsilon_2 + 12 c_2 \epsilon_1^2 - 6 c_{3A} \epsilon_1 + 6 c_1 \epsilon_{3A}, \\
c'_{4B} &= c_{4B} + 32 c_1 \epsilon_1^3 - 24 c_1 \epsilon_1 \epsilon_2 + 12 c_2 \epsilon_1^2 - 2(c_{3A} + 4 c_{3B})\epsilon_1 + 2 c_1(\epsilon_{3A} + 4 \epsilon_{3B}), \\
c'_{4C} &= c_{4C} + 32 c_1 \epsilon_1^3 - 24 c_1 \epsilon_1 \epsilon_2 + 8 c_2 \epsilon_1^2 - 2(2 c_{3A} + c_{3C})\epsilon_1 + 2 c_1(2 \epsilon_{3A} + \epsilon_{3C}), \\
c'_{4D} &= c_{4D} + 32 c_1 \epsilon_1^3 - 24 c_1 \epsilon_1 \epsilon_2 + 8 c_2 \epsilon_1^2 - 2(4 c_{3B} + c_{3C})\epsilon_1 + 2 c_1(4 \epsilon_{3B} + \epsilon_{3C}), \\
c'_{4E} &= c_{4E} - 8 c_2 \epsilon_1^2 + 4(c_{3A} - c_{3C})\epsilon_1 - 4 c_1(\epsilon_{3A} - \epsilon_{3C}), \\
c'_{4F} &= c_{4F} - 4 c_2 \epsilon_1^2 + 2(2 c_{3B} - c_{3C})\epsilon_1 - 2 c_1(2 \epsilon_{3B} - \epsilon_{3C}), \\
c'_{4G} &= c_{4G},
\end{aligned} \tag{3.16}$$

and

$$c'_{4H} = c_{4H} + Y, \quad c'_{4I} = c_{4I} + 2Y, \quad c'_{4J} = c_{4J} - Y, \quad Y = 4(c_1 \epsilon_{3D} - c_{3D} \epsilon_1). \tag{3.17}$$

In consequence of (3.16) 4 loop scheme invariants are given, for the planar graphs, by

$$c_{4A} - c_{4B} - c_{4C} + c_{4D}, \quad 2 c_{4A} - 2 c_{4C} + c_{4E}, \quad c_{4A} + c_{4B} - 2 c_{4C} + 2 c_{4F}, \quad c_{4G}, \tag{3.18}$$

and for the non planar ones from (3.17)

$$2 c_{4H} - c_{4I}, \quad c_{4H} + c_{4J}. \tag{3.19}$$

The results of [4]<sup>1</sup> give scheme invariants which may be expressed in terms of a basis given by

$$\begin{aligned}
c_{4A} - c_{4B} - c_{4C} + c_{4D} &= 0, \\
2 c_{4A} - 2 c_{4C} + c_{4E} &= -\frac{1}{2}, \quad c_{4A} + c_{4B} - 2 c_{4C} + 2 c_{4F} = \frac{1}{2}\zeta(3) - \frac{1}{2}, \quad c_{4G} = -\frac{5}{2}, \\
2 c_{4H} - c_{4I} &= 0, \quad c_{4H} + c_{4J} = -3\zeta(3), \quad c_{4K} = -10\zeta(5).
\end{aligned} \tag{3.20}$$

Using the freedom allowed by (3.16) and the results in [4] there is a minimal scheme where we choose

$$c_{4A} = c_{4B} = c_{4C} = c_{4D} = c_{4H} = c_{4I} = 0, \tag{3.21}$$

with only  $c_{4E}$ ,  $c_{4F}$ ,  $c_{4G}$ ,  $c_{4J}$ ,  $c_{4K}$ , given by (3.20), non zero.

## 4 Conclusion

Although the results of this paper are rather technical it is important to understand the consequences of the freedom of the choice of scheme. In the case of  $\mathcal{N} = 1$  supersymmetry the NSVZ expression for the gauge  $\beta$ -function, expressed in terms of the anomalous

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<sup>1</sup>With our notation their individual results calculated with a minimal subtraction scheme, if not quoted in the text, are  $c_{4A} = c_{4B} = \frac{1}{8}(2\zeta(3) - 1)$ ,  $c_{4C} = c_{4D} = \frac{1}{3}$ ,  $c_{4E} = \frac{1}{12}(5 - 6\zeta(3))$ ,  $c_{4F} = \frac{5}{24}$  and for the non planar graphs  $2c_{4H} = c_{4I} = -\frac{1}{2}(6\zeta(3) - 3\zeta(4))$ ,  $c_{4J} = -\frac{3}{4}(6\zeta(3) - 3\zeta(4))$ . Corresponding results for  $c_{4B} + c_{4D}$  and  $c_{4A} + 2c_{4C} + c_{4E} + c_{4F} + c_{4G} + c_{4J}$  as well as  $c_{4I} = 2c_{4H}$ ,  $c_{4K}$  can be read off from [5].

dimension  $\gamma$ , presupposes a particular choice of scheme [6, 7]. Results stemming from the  $a$ -theorem also constrain  $\gamma$  in terms of lower order contributions, [8], [9], [2], [10]. The equations used to obtain such results are not manifestly scheme independent but [9] were able to determine, for chiral matter fields belonging to a gauge representation  $R$ , the scheme independent contributions  $(C_R g^2)^p$ ,  $t_a t_a = C_R \mathbb{1}$ , to  $\gamma$  for these matter fields for any  $p$  from the expansion of  $(1 - (1 + 8 g^2 C_R)^{\frac{1}{2}})/2$ . The results quoted above for the scheme independent  $c_1$ ,  $c_2$ ,  $c_{3C}$ ,  $c_{4G}$  can be fitted by the coefficients in the expansion of  $((1 + 4x)^{\frac{1}{2}} - 1)/4$ , but there is no derivation of such results to all orders as yet<sup>2</sup>. At any order diagrams with a new topology, such as those corresponding to  $c_{3D}$ ,  $c_{4K}$ , so that the contribution to  $\text{tr}(\gamma)$  is given by a symmetric graph, give rise to new transcendental numbers and so are clearly independent. The presence of  $\zeta(3)$  in the scheme invariant in (3.20) containing  $c_{4F}$  involving four loop planar graphs suggests, assuming the results in [5], [4] are correct, that there is no simple recursive formula for the coefficients of all graphs with the same basic topology.

One issue which merits further discussion is the role of anomalies since the irrelevance of antisymmetric contributions to the anomalous dimension depends on all global symmetries of the kinetic term being non anomalous. Of course with chiral fermions this need not be true.

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<sup>2</sup>Extending the discussion in [2] gives for the planar diagrams  $2(2c_{4A} - 2c_{4C} + c_{4E}) - c_{4G} = \frac{3}{2}$ . The results in (3.20) for  $c_{4H}$ ,  $c_{4I}$ ,  $c_{4J}$  may also be obtained as consistency conditions in terms of  $c_{3D}$ .

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